Integration of nonlinear Partial Differential Equations by using matrix algebraic systems

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Abstract

The paper develops the method for construction of families of particular solutions to some classes of nonlinear Partial Differential Equations (PDE). Method is based on the specific link between algebraic matrix equations and PDE. Admittable solutions involve arbitrary functions of either single or several variables.

1 Introduction

Analysis of nonlinear Partial Differential Equations (PDE) is severe problem in mathematical physics. Many different methods have been developed for analytical investigation of nonlinear PDE during last decades: Inverse Scattering Problem [1, 2, 3, 4, 5, 6, 7], Sato theory [8, 9, 10, 11], Hirota bilinear method [12, 13, 14, 15], Penlevé method [16, 17, 18], $\bar{\partial}$-problem [19, 20, 21], with some generalizations [22, 23, 24, 25, 26], quasi-classical version of the $\bar{\partial}$-problem [27, 28, 29]. A wide class of PDE (so-called completely integrable

systems) has been studied better then others. Nevertheless, there are many methods which work in nonintegrable case as well: [12, 13, 14, 15, 16, 17, 18].

We represent the method for construction of the families of particular solutions to some classes of multidimensional nonlinear PDE. This method is based on general properties of linear algebraic matrix equations. Essentially we develop some ideas represented in the ref.[8] and recently in the ref. [11].

General algorithm is discussed in the Sec. 2. We consider systems, admitting solutions depending on arbitrary functions of either single or several variables. Sec. 3 represents some examples among which are Kadomtsev-Petviashvili equation (KP) and Devi-Stewartson equation (DS). Conclusions are given in the Sec.5.

2 General results

The algorithm represented in this section is based on the fundamental properties of linear matrix algebraic equation, which is written in the following form:

$$\Psi U = \Phi, \tag{1}$$

where $\Psi = \{\psi_{ij}\}$ is $N \times N$ matrix, U and Φ are $N \times M$ matrices. Let us recall these properties.

1. If Ψ is nondegenerate matrix, i.e.

   $$\det \Psi \neq 0, \tag{2}$$

   then equation (1) has unique solution which can be written in the following form:

   $$U = \Psi^{-1}\Phi. \tag{3}$$

   Only nondegenerate matrices Ψ will be considered hereafter.

2. (*consequence of the previous property*) If $\Phi = 0$ and condition (2) is held, then the equation (1) has only the trivial solution

   $$U \equiv 0. \tag{4}$$

3. (*superposition principle*) Consider the set of K matrix equations with the same matrix Ψ:

$$\Psi U_i = \Phi_i, \quad i = 1, \dots, K. \tag{5}$$

Then for any set of scalars b_k ($k = 1, \dots, K$), function $\tilde{U} = \sum_{k=1}^{K} b_k U_k$ is solution of the following matrix equation

$$\Psi \tilde{U} = \sum_{k=1}^{K} b_k \Phi_k. \tag{6}$$

4. (*consequence of properties 2 and 3*) If columns Φ_i are linearly dependent, i.e there are scalars a_k, $k = 1, \dots, K$, such that

$$\sum_{k=1}^{K} a_k \Phi_k = 0, \tag{7}$$

then

$$\sum_{k=1}^{K} a_k U_k = 0. \tag{8}$$

Note that analogous properties of linear integral equation have been used in the classical dressing method based on the $\bar{\partial}$-problem [19, 20, 21].

We will use two sets of variables, which will be introduced in the functions Φ and Ψ: $x = (x_1, \dots, x_Q)$, $t = (t_1, t_2, \dots)$, where $Q = \dim(x)$. The next statement follows from the above properties of the linear equations.

If there is transformation T, which maps the nonhomogeneous equation (1) into the homogeneous

$$\Psi \tilde{U}(U) = 0, \tag{9}$$

then

$$\tilde{U}(U) = 0, \tag{10}$$

or

$$\Phi = \Psi U \ \stackrel{T}{\to} \ 0 = \Psi \tilde{U}(U) \Longrightarrow \tilde{U}(U) \equiv 0 \tag{11}$$

If $\tilde{U}$ depends on U and its derivatives, then equation (10) represents the matrix PDE for U.

We will see that transformation T is not unique. One has the manifold of transformations $\mathbb{T}$

$$T_j \in \mathbb{T}: \ \Phi = \Psi U \ \stackrel{T_j}{\longrightarrow} \ 0 = \Psi \tilde{U}_j(U) \Longrightarrow \tilde{U}_j(U) \equiv 0 \tag{12}$$

which is uniquely defined by the equations introducing variables x and t in the matrices Φ and Ψ. Namely,

$$\Psi_{x_k} = \Psi B_k + \Phi C_k, \ \ k = 1, \dots, Q \tag{13}$$

(B_k and C_k are $N \times N$ and $M \times N$ matrices respectively) and

$$M^n_\alpha \Psi = 0, \quad M^n_\alpha \Phi = 0, \ \ M^n_\alpha = \partial_{t_\alpha} + \partial^\alpha, \ \ \sum_{k=1}^{Q} \alpha_k = n, \tag{14}$$

where n is order of differential operator. We use Q-dimensional vector subscript $\alpha = (\alpha_1, \dots \alpha_Q)$ and notation $\partial^\alpha = \prod_{j=1}^{Q} \partial_{x_j}^{\alpha_j}$. Systems (13) and (14) should be compatible, which leads to relations among matrices B_j and C_j (see the next subsection).

Each operator M^n_α defines transformation $T_\alpha \in \mathbb{T}$. In fact, let us apply M^n_α to both sides of the eq. (1) and use eqs. (14):

$$\begin{aligned} 0 = M^n_\alpha \Phi = (\partial_{t_\alpha} \Psi) U + \Psi \partial_{t_\alpha} U + \partial^\alpha (\Psi U) = -(\partial_\alpha \Psi) U + \Psi \partial_{t_\alpha} U + \\ \sum_{j_1=0}^{\alpha_1} \cdots \sum_{j_Q=0}^{\alpha_Q} \left(\prod_{k=1}^{Q} C_{\alpha_k}^{j_k} \right) \left(\prod_{n=1}^{Q} \partial_{x_n}^{j_n} \Psi \right) \left(\prod_{m=1}^{Q} \partial_{x_m}^{\alpha_m - j_m} U \right), \end{aligned} \tag{15}$$

$C^k_n = \dfrac{n!}{k!(n-k)!}$ are binomial coefficients. Due to the eqs.(1) and (13) , one has the following expression for $\partial_{x_j} \Psi$:

$$\partial_{x_j} \Psi = \Psi V_{e^j}, \quad V_{e^j} = B_j + U C_j, \ \ e^j = (\underbrace{0, \dots, 0}_{j-1}, 1, \underbrace{0, \dots, 0}_{Q-j}). \tag{16}$$

Analogously we can write

$$\partial^\beta \Psi = \Psi V_\beta, \tag{17}$$

where V_β can be found recursively. In fact, one has

$$\begin{aligned}\partial^\beta \Psi = \partial_{x_j}\partial_{(\beta_1,\dots,\beta_j-1,\dots,\beta_Q)}\Psi = \partial_{x_j}(\Psi V_{(\beta_1,\dots,\beta_j-1,\dots,\beta_Q)}) = \\ (\partial_{x_j}\Psi)V_{(\beta_1,\dots,\beta_j-1,\dots,\beta_Q)} + \Psi(\partial_{x_j}V_{(\beta_1,\dots,\beta_j-1,\dots,\beta_Q)}) = \\ \Psi\left(V_{e^j}V_{(\beta_1,\dots,\beta_j-1,\dots,\beta_Q)} + \partial_{x_j}V_{(\beta_1,\dots,\beta_j-1,\dots,\beta_Q)}\right)\end{aligned} \tag{18}$$

i.e.

$$\begin{aligned}V_\beta = V_{e^j}V_{(\beta_1,\dots,\beta_j-1,\dots,\beta_Q)} + \partial_{x_j}V_{(\beta_1,\dots,\beta_j-1,\dots,\beta_Q)}, \\ V_0 = V_{\underbrace{(0,\dots,0)}_{Q}} = I_{N,N}.\end{aligned} \tag{19}$$

Owing to the eq.(17) we can replace factor $\left(\prod_{n=1}^Q \partial_{x_n}^{j_n}\Psi\right)$ in (15) with $\Psi V_{(j_1,\dots,j_Q)}$, which results in

$$\begin{aligned}0 = -(\partial_\alpha\Psi)U + \Psi\partial_{t_\alpha}U + \\ \Psi\sum_{j_1=1}^{\alpha_1}\cdots\sum_{j_Q=1}^{\alpha_Q}\left(\prod_{k=1}^Q C_{\alpha_k}^{j_k}\right)V_{(j_1,\dots,j_Q)}\prod_{m=1}^Q \partial_{x_m}^{\alpha_m-j_m}U = \Psi\tilde{U}_\alpha,\end{aligned} \tag{20}$$

$$\tilde{U}_\alpha = \partial_{t_\alpha}U + {\sum_{j_1,\dots j_Q}}'\left(\prod_{k=1}^Q C_{\alpha_k}^{j_k}\right)V_{(j_1,\dots,j_Q)}\prod_{m=1}^Q \partial_{x_m}^{\alpha_m-j_m}U, \tag{21}$$

where ${\sum_{j_1,\dots,j_Q}}'$ means the sum over all values $j_1,\dots,j_Q$ such that $(\sum_{m=1}^Q j_m) <$ n The associated system of nonlinear PDE has the form (see (10))

$$\partial_{t_\alpha}U + {\sum_{j_1,\dots j_Q}}'\left(\prod_{k=1}^Q C_{\alpha_k}^{j_k}\right)V_{(j_1,\dots,j_Q)}\prod_{m=1}^Q \partial_{x_m}^{\alpha_m-j_m}U = 0 \tag{22}$$

2.1 Compatibility of the system (13) and (14)

Studying the compatibility condition for the system (13) we concentrate on two particular cases

1. compatibility condition produces the additional differential equation for the function U and $Q-1$ differential equations for the function φ. In this case ϕ (and U) involve N arbitrary scalar functions of single variable.

2. compatibility condition produces only the differential equation for the function U. In this case ϕ (and U) involve $N \times M$ arbitrary scalar functions of Q variables.

Intermediate case (i.e. φ and U depend on arbitrary functions of P ($1 < P < Q$) variables) is also possible, but it is not regarded in this paper.

2.1.1 Matrix U depends on N arbitrary scalar functions of single variable

We start the analysis with splitting the eq. (13) into two equations by using the following structure of the matrices:

$$\Psi = \left[\begin{array}{c|c} \varphi & \chi \end{array} \right], \quad B_j = \left[\begin{array}{c|c} b_j & B_{1j} \\ \hline 0_{N-M,M} & B_{2j} \end{array} \right], \tag{23}$$

$$C_j = \left[\begin{array}{c|c} C_{0j} & C_{1j} \end{array} \right], \quad C_{01} = I_M, \quad j = 1, \dots, Q \tag{24}$$

where φ is $N \times M$, χ is $N \times (N-M)$; b_j, C_{0j} are $M \times M$, B_{1j} and C_{1j} are $M \times (N-M)$; B_{2j} are $(N-M) \times (N-M)$ matrices for all $j = 1, \dots, Q$; hereafter $0_{A,B}$ with arbitrary A and B is $A \times B$ matrix of zeros; I_M is $M \times M$ identity matrix. Having matrices with given structure one can split the system (13) as follows:

$$\varphi_{x_j} = \varphi b_j + \Phi C_{0j}, \tag{25}$$

$$\chi_{x_j} = \chi B_{2j} + \varphi B_{1j} + \Phi C_{1j} \tag{26}$$

The eq.(25) with $j = 1$ defines Φ in terms of φ (remember, that $C_{01} = I_M$):

$$\Phi = \varphi_{x_1} - \varphi b_1, \tag{27}$$

Substituting this expression into the eq.(25) with $j > 1$ one gets the overdetermined system of equations for φ :

$$\varphi_{x_j} = \varphi_{x_1} C_{0j} + \varphi(b_j - b_1 C_{0j}), \quad j > 1. \tag{28}$$

Compatibility of this system with different j leads to the following relations among matrices:

$$\begin{aligned} [C_{0k}, C_{0j}] = 0, \\ [b_j, C_{0k}] - [b_k, C_{0j}] + C_{0k} b_1 C_{0j} - C_{0j} b_1 C_{0k} = 0, \\ [b_j - b_1 C_{0j}, b_k - b_1 C_{0k}] = 0. \end{aligned} \tag{29}$$

From another point of view, eqs. (1), (13) and (25) lead to the additional differential equation for U:

$$\begin{aligned} E_M[b_i, b_j] + UC_{0i}b_j + U_{x_i}C_{0j} + B_iUC_{0j} + UC_iUC_{0j} = \\ UC_{0j}b_i + U_{x_j}C_{0i} + B_jUC_{0i} + UC_jUC_{0i}, \end{aligned} \tag{30}$$

where E_M is matrix,composed of first M column of the N-dimensional identity matrix I_N. So, eq. (25) describes one more transformation $T^{\mathrm{comp}} \in \mathbb{T}$.

The system (26) is overdetermined system defining χ in terms of φ. Its compatibility condition leads to the following relations

$$\begin{aligned} A_3\chi + A_2\varphi_{x_1x_1} + A_1\varphi_{x_1} + A_0\varphi = 0, \\ A_3 = [B_{2i}, B_{2j}] \equiv 0 \\ A_2 = C_{0i}C_{1j} - C_{0j}C_{1i} \equiv 0, \\ A_n = A_n(C_{0i}, C_{0j}, C_{1i}, C_{1j}, B_j) \equiv 0, \quad n = 0, 1. \end{aligned} \tag{31}$$

We don't represent expressions for A_0 and A_1, because they are too complicated. But in the particular case, when $b_j = 0_{M,M}$ and $C_{1j} = 0_{M,N-M}$ the above system has simpler form

$$\begin{aligned} [B_{2i}, B_{2j}] = 0 \\ C_{0i}C_{1j} - C_{0j}C_{1i} = 0, \\ C_{1i}B_{2j} + C_{0i}B_{1j} - C_{1j}B_{2j} - C_{0j}B_{1i} = 0 \\ B_{1i}B_{2j} - B_{1j}B_{2i} = 0. \end{aligned} \tag{32}$$

To satisfy both systems (13) and (14) φ should be written in the form:

$$\varphi = \int_{-\infty}^{\infty} c(k_1) \exp\left[\sum_{i=1}^{Q} k_i x_i + \sum_{\alpha} \omega_\alpha t_\alpha\right] dk_1, \tag{33}$$

where $c(k_1)$ ($N \times M$ matrix function of argument) and k_j satisfy the dispersion relation associated with eq.(28), ω_α satisfy the dispersion relation for eq.(14). One can see that the above expression for φ (and U due to the eq. (3)) involves N arbitrary functions of single variable. Nonlinear equations generated by the operators M_α^n have the form (22)

2.1.2 Matrix U depending on $N \times M$ arbitrary scalar functions of Q variables

In this section we consider matrix equation (1) in the form

$$\Phi = \Psi U, \;\; \Phi = \left[\begin{array}{ccc} \Phi_1 & \cdots & \Phi_P \end{array}\right], \;\; U = \left[\begin{array}{ccc} U_1 & \cdots & U_P \end{array}\right], \;\; \text{or} \tag{34}$$

$$\Phi_k = \Psi U_k, \;\; k = 1, \ldots, P, \tag{35}$$

where Φ_j and U_j are $N \times M$, Ψ is $N \times N$ matrices. Let matrix C_k be of the form:

$$C_k = \left[\begin{array}{c} 0_{M(k-1),N} \\ \tilde{C}_k \\ 0_{N-Mk,N} \end{array}\right], \tag{36}$$

where $\tilde{C}_k$ is $M \times N$ matrix. So that eq.(13) can be written as follows:

$$\Psi_{x_k} = \Psi B_k + \Phi_k \tilde{C}_k, \;\; k = 1, \ldots, Q, \tag{37}$$

variables t_n are introduced by the same eq.(14). Assume the following structure of matrices:

$$\Psi = \left[\begin{array}{c|c} \varphi & \chi \end{array}\right], \;\; \tilde{C}_k = \left[\begin{array}{c|c} I_M & C_{1k} \end{array}\right], \;\; B_k = \left[\begin{array}{c|c} b_k & B_{1k} \\ \hline 0_{M,N-M} & B_{2k} \end{array}\right], \tag{38}$$

where b_k, are $M \times M$ matrices, C_{1k} and B_{1k} are $M \times (N-M)$ matrices, B_{2k} are $(N-M) \times (N-M)$ matrices for all $k = 1, \ldots, Q$. Now we can split the eq.(37) into the following pair of equations

$$\varphi_{x_k} = \varphi b_k + \Phi_k, \tag{39}$$

$$\chi_{x_k} = \chi B_{2k} + \varphi B_{1k} + \Phi_k C_{1k}. \tag{40}$$

Equation (39) defines function Φ_k in terms of φ. At the same time, Φ_k can be eliminated from the eq.(39) due to the eq.(35), so that eq.(39) represents overdetermined system for φ with compatibility condition (compare with eq.(30))

$$\begin{aligned} E_M[b_i, b_j] + U^j_{x_i} + U^i b_j + B_i U^j + U^i \tilde{C}_i U^j = \\ U^i_{x_j} + U^j b_i + B_j U^i + U^j \tilde{C}_j U^i, \end{aligned} \tag{41}$$

So, additional transformation $T^{\text{comp}} \in \mathbb{T}$ is based on the system (39).

Compatibility of the system (40) gives relations among matrices B_j and C_j: for all j and k

$$\begin{aligned} C_{1j} - C_{1k} &= 0, \\ [B_{2j}, B_{2k}] &= 0, \\ B_{1j} - b_j C_{1j} + C_{1j} B_{2j} &= 0. \end{aligned} \tag{42}$$

Expression for φ satisfying both systems (13) and (14) is following:

$$\varphi = \int_{-\infty}^{\infty} c(k_1, \dots, k_Q) \exp\left[\sum_{i=1}^{Q} k_i x_i + \sum_{\alpha} \omega_\alpha t_\alpha\right] dk_1 \dots dk_Q, \tag{43}$$

where c is $N \times M$ arbitrary matrix function of arguments; ω_α satisfy the dispersion relations for (14). Due to the formula (3) and (43), U depends on $N \times M$ arbitrary functions of Q variables. Nonlinear equations generated by the operators M_α^n keep the form (22)

3 Examples

We consider two hierarchies of PDE, admitting solutions depending on N functions of single variables (see subsection 2.1.1). Inside of them are classical KP and DS hierarchies. It is important to note that N can be arbitrary integer for later, which is not true in general.

3.1 KP hierarchy

Let $M = 1$, $Q = 1$, $M^n = \partial_{t_n} + \partial_x^n$, $n = 2, 3, \dots$. Operators M^n generate the following hierarchy (see eq.(22)):

$$\begin{gathered} U_{t_n} + \sum_{i=1}^{n} C_n^i V_{n-i} \partial_x^i U = 0, \\ V_0 = I_N, ; \; V_1 = B + UC, \;\; V_n = V_1 V_{n-1} + \partial_x V_{n-1}, \end{gathered} \tag{44}$$

where $B \equiv B_1$ and $C \equiv C_1$. In this case one has only one eq.(13) and no compatibility condition for it, i.e. matrices B and C are *arbitrary* matrices

having structure (23) and (24). If

$$B = \left[\begin{array}{c|cccc} 0 & 1 & 0 & 0 & \dots \\ \hline 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right], \quad C = \left[\begin{array}{c|ccc} 1 & 0 & 0 & \dots \end{array}\right], \quad U = \left[\begin{array}{c} u_1 \\ \vdots \\ u_N \end{array}\right], \tag{45}$$

N is arbitrary, then system (44) represent KP hierarchy [8], . KP can be written for the function $u \equiv u_{1x}$ using eq.(44) with $n = 2, 3$:

$$u_t + \frac{1}{4}u_{xxx} + \frac{3}{4}\partial_x^{-1}u_{yy} + 3uu_x = 0 \tag{46}$$

3.2 DS hierarchy

Let $M = 2$, $Q = 2$, $x_1 = x$, $x_2 = y$. Consider operators $M^n = \partial_{t_n} + \partial_x^n$. This operators generate the same hierarchy (44) with appropriate dimensions for matrix U,

$$V_1 = B_1 + UC_1, \tag{47}$$

and additional nonlinear equation (30):

$$U_x C_{02} + B_1 U C_{02} + UC_1 U C_{02} = U_y C_{01} + B_2 U C_{01} + UC_2 U C_{01} \tag{48}$$

with relations among matrices B_j and C_j given by (29) and (31) (or (32)). DS hierarchy corresponds to the following choice of matrices B_j and C_j:

$$B_1 = \left[\begin{array}{cc|cccc} 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ \hline 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right], \quad B_2 = \left[\begin{array}{cc|ccccc} 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right], \tag{49}$$

$$C_1 = \left[\begin{array}{cc|cc} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \end{array}\right], \quad C_2 = \left[\begin{array}{cc|cc} 0 & 1 & 0 & \dots \\ 1 & 0 & 0 & \dots \end{array}\right], \quad U = \left[\begin{array}{cc} u_1 & v_1 \\ \vdots & \vdots \\ u_N & v_n \end{array}\right],$$

N is arbitrary. It is simply to check that conditions (29,32) are satisfied. Let us consider the eq.(44) with $n = 2$ together with eq.(48). The following change of variables

$$u_1 = \frac{1}{4}(r - q + 2w_x), \quad u_2 = \frac{1}{4}(r + q + 2w_y), \tag{50}$$
$$v_1 = \frac{1}{4}(-r - q + 2w_y), \quad v_2 = \frac{1}{4}(-r + q + 2w_x).$$

transforms them into

$$r_{t_2} - r_{xy} - 2rw_{xy} = 0, \quad q_{t_2} + q_{xy} + 2qw_{xy} = 0, \quad w_{xx} - w_{yy} = qr, \tag{51}$$

which results in DS after reduction $r = \psi$, $q = \bar{\psi}$, $t_2 = it$, where $i^2 = -1$, bar means complex conjugated value

4 Conclusions

The suggested version of the dressing method is one more form of representation of the systems of nonlinear PDE, which admit an infinite number of commuting flows, generated by the operators M^n_α. Among them are classical completely integrable systems of equations. The feature of these system is that their solutions admit *any* number of arbitrary scalar functions of single variable (N is arbitrary for them). One can see that there is another type of systems, which admit an infinite number of commuting flows, while N is given. We don't know interesting examples for this case. We also have not found the systems, whose solutions would admit any number of arbitrary scalar functions of several variables: for all nontrivial examples this number has to be fixed. These aspects will be studied later.

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